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# Linear quantum transformation and normal product calculation of boson exponential quadratic operators*

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**Abstract.** In this work, we present a general approach for deriving the normal product form of multi-mode boson exponential quadratic operators from the viewpoint of linear quantum transformations in boson Fock space. The simplicity and generality of our formula are shown by some examples.

## 1. Introduction

Quantum transformation plays an important role in understanding and solving many problems in quantum mechanics. The original works in this field were performed by Bogoliubov [1] and Valatin [2] and, after these, transformation theory achieved many further developments [3, 4].

Balian and Brezin [5] presented the non-unitary Bogoliubov transformation theory in 1969 and, recently, Zhang (one of the authors of this work) and Tang [6, 7] and Yu [8] offered a general theory of linear quantum transformation (LQT) in Fock space. In all of these theories, the only assumption is that the commutation rules *but not the Hermitian relation* of operators are preserved after transformation. In [5], the exponential quadratic form of the transformation operator for transformation matrix $e^M$ is presented, where $M$ is a $2n \times 2n$ complex matrix and $e^M$ is an element of the sympletic group. In [6], the normal product exponential quadratic form of the transformation operator for the corresponding symplectic $2n \times 2n$ partitioned matrix is presented

$$M = \begin{pmatrix} A & D \\ \tilde{B} & \tilde{C} \end{pmatrix}.$$

However, none of these papers has shown a general approach to deriving the normal product form of the exponential (complete) quadratic operator (EQO).

The EQO is widely used in quantum mechanics and quantum optics, such as the time evolution operator $e^{-iHt}$ and various transformation and state operators, etc. Usually, it provides us with a convenient method if the operator is transformed into its normal product form. Some special forms of the EQO have been transformed into normal product forms by

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the so-called 'IWOP' technique in [9], but no general formula was obtained in that work. Here, we present a general approach to transform a multi-mode boson EQO into normal product form from the LQT viewpoint.

This paper is organized as follows. In section 2, we introduce the LQT theory in short and obtain a general formula which relates an EQO and its normal product form from the LQT viewpoint. In section 3, we give some applications of this formula. Through these applications, the generality and simplicity of our approach can be seen. Before the theoretical statement, we give our notation in $n$-mode boson Fock space. We postulate a fundamental operator as follows

$$\Lambda = (a^+, \tilde{a}) \tag{1}$$

where $a^+$ and $\tilde{a}$ are $n$-mode boson creation and annihilation operators, respectively.

$$a^+ = (a_1^+, a_2^+, \dots, a_n^+) \qquad \tilde{a} = (a_1, a_2, \dots, a_n). \tag{2}$$

Here the notation '~' is the transpose of $n$-dimensional space. These operators satisfy the following boson commutation rules:

$$[a_i, a_j^+] = \delta_{ij} \qquad [a_i, a_j] = [a_i^+, a_j^+] = 0. \tag{3}$$

We can rewrite equation (3) in the equivalent form

$$[\tilde{\Lambda}, \Lambda] = \Sigma^{-1} \tag{4}$$

where

$$\Sigma = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{5}$$

and where $I$ is the $n$-dimensional unit matrix.

## 2. LQT theory and the transformation between EQO and its normal product form in $n$-mode boson Fock space

Let us consider the following linear quantum transformation in $n$-mode boson Fock space:

$$\Lambda' = U\Lambda U^{-1} = \Lambda \cdot M \tag{6}$$

where $M \in C^{2n\times 2n}$ and is called the LQT matrix and $U$ is a bosonic operator cluster and is called an LQT operator. $\Lambda'$ still satisfies the boson commutation rule

$$[\tilde{\Lambda}', \Lambda'] = \Sigma^{-1}. \tag{7}$$

In fact, this is the only assumption for the quantum transformation. This assumption demands that $M \in Sp(2n, c)$ [5, 7], i.e.

$$M\Sigma\tilde{M} = \Sigma. \tag{8}$$

From equation (20a) of [5], the LQT operator corresponding to matrix $M$ is

$$U = \exp(\tfrac{1}{2}\Lambda \ln M\Sigma\tilde{\Lambda}). \tag{9}$$

On the other hand, from equation (42) of [5], or from [7], we can immediately obtain

$$U' =: \exp\left[\frac{1}{2}\Lambda \begin{pmatrix} -C^{-1}\tilde{D} & C^{-1}-1 \\ \tilde{C}^{-1}-1 & BC^{-1} \end{pmatrix} \tilde{\Lambda}\right]: \tag{10}$$

which also corresponds to the matrix $M = \begin{pmatrix} A & D \\ \tilde{B} & \tilde{C} \end{pmatrix}$. We know that the corresponding LQT operator is determined up to a $c$ number [7] as $M$ is given, so that we have

$$U = c \cdot U'. \tag{11}$$

In fact, without any loss of generality, any $\Omega$ of boson EQO can be written in the following form:

$$\Omega = \exp\left[\frac{1}{2}\Lambda \begin{pmatrix} A_1 & A_3 \\ \tilde{A}_3 & A_2 \end{pmatrix} \tilde{\Lambda}\right] = \exp\left[\frac{1}{2}\Lambda \begin{pmatrix} A_3 & -A_1 \\ A_2 & -\tilde{A}_3 \end{pmatrix} \Sigma\tilde{\Lambda}\right]$$

where $A_1 = \tilde{A}_1$, $A_2 = \tilde{A}_2$, or we have

$$\Omega = \exp\left\{\frac{1}{2}\Lambda \ln\left[\exp\begin{pmatrix} A_3 & -A_1 \\ A_2 & -\tilde{A}_3 \end{pmatrix}\right] \Sigma\tilde{\Lambda}\right\}. \tag{12}$$

It is easy to see that

$$\Sigma^{-1} \exp\begin{pmatrix} A_3 & -A_1 \\ A_2 & -\tilde{A}_3 \end{pmatrix} \Sigma = \exp\left[-\widetilde{\begin{pmatrix} A_3 & -A_1 \\ A_2 & -\tilde{A}_3 \end{pmatrix}}\right]$$

i.e. $\exp\begin{pmatrix} A_3 & -A_1 \\ A_2 & -\tilde{A}_3 \end{pmatrix}$ is an element of $Sp(2n, c)$.

Therefore, we can regard any EQO as a LQT operator corresponding to the LQT matrix

$$M' = \exp\begin{pmatrix} A_3 & -A_1 \\ A_2 & -\tilde{A}_3 \end{pmatrix} = \begin{pmatrix} A' & D' \\ \tilde{B}' & \tilde{C}' \end{pmatrix}. \tag{13}$$

According to equations (8)–(10), we get the following formula:

$$\Omega = \exp\left[\frac{1}{2}\Lambda \begin{pmatrix} A_3 & -A_1 \\ A_2 & -\tilde{A}_3 \end{pmatrix} \Sigma\tilde{\Lambda}\right] = c : \exp\left[\frac{1}{2}\Lambda \begin{pmatrix} -C'^{-1}\tilde{D}' & C'^{-1}-1 \\ \tilde{C}'^{-1}-1 & B'C'^{-1} \end{pmatrix} \tilde{\Lambda}\right] : . \tag{14}$$

The $c$ number is determined in the appendix as

$$c = (\det C')^{-1/2}. \tag{15}$$

Equations (13)–(15) are our formulae for transforming an EQO into its normal product form. To derive an EQO's normal product form using these equations we only have to transform the matrix

$$\exp\begin{pmatrix} A_3 & -A_1 \\ A_2 & -\tilde{A}_3 \end{pmatrix}$$

into its partitioned form

$$M' = \begin{pmatrix} A' & D' \\ \tilde{B}' & \tilde{C}' \end{pmatrix}.$$

In fact, equation (14) is in agreement with equation (42) of [5] except that there the constant number $c$ is neglected. The above results can be easily extended to the cases which contain linear terms in LQT operators. One can take the following steps:

$$\Omega = \exp\left[\frac{1}{2}\Lambda \begin{pmatrix} A_1 & A_3 \\ \tilde{A}_3 & A_2 \end{pmatrix} \tilde{\Lambda} + (k_1^+, \tilde{l}_1)\tilde{\Lambda}\right] = \exp\left[\frac{1}{2}\Lambda_1 \begin{pmatrix} A_1 & A_3 \\ \tilde{A}_3 & A_2 \end{pmatrix} \tilde{\Lambda}_1 - c'\right]$$

where $\Lambda_1 = (\bar{b}, \tilde{b})$ and $\bar{b} = a^+ + k^+, \tilde{b} = \tilde{a} + \tilde{l}$, provided that $k^+$ and $\tilde{l}$ satisfy

$$(k^+ A_1 + \tilde{l}\tilde{A}_3) = k_1^+ \qquad (\tilde{l}A_2 + k^+ A_3) = \tilde{l}_1$$

respectively, and

$$c' = \frac{1}{2}(k^+, \tilde{l}) \begin{pmatrix} A_1 & A_3 \\ \tilde{A}_3 & A_2 \end{pmatrix} \begin{pmatrix} \tilde{k}^+ \\ l \end{pmatrix}.$$

The normal product form can be obtained from formula (14) by substituting $\Lambda$ with $\Lambda_1$. Thus, in the following, we consider only the pure quadratic cases.

## 3. Some applications

Now we transform some EQOs into their normal product form using formulae (13) and (14) of the above section

*(i)* $e^{X\Gamma\tilde{X}}$ *and* $e^{P\Gamma\tilde{P}}$. Here $\Gamma$ is a $3 \times 3$ symmetric complex matrix, $X = \frac{1}{\sqrt{2}}(\tilde{a} + a^+)$ and $P = \frac{-i}{\sqrt{2}}(\tilde{a} - a^+)$. We can transform the two operators into Fock space as

$$e^{X\Gamma\tilde{X}} = \exp\left[\frac{1}{2}\Lambda \begin{pmatrix} \Gamma & -\Gamma \\ \Gamma & -\Gamma \end{pmatrix} \Sigma\tilde{\Lambda}\right] \tag{16}$$

$$e^{P\Gamma\tilde{P}} = \exp\left[\frac{1}{2}\Lambda \begin{pmatrix} \Gamma & \Gamma \\ -\Gamma & -\Gamma \end{pmatrix} \Sigma\tilde{\Lambda}\right]. \tag{17}$$

Noticing that $\Gamma = \tilde{\Gamma}$ and

$$\exp\begin{pmatrix} \Gamma & -\Gamma \\ \Gamma & -\Gamma \end{pmatrix} = \begin{pmatrix} 1+\Gamma & -\Gamma \\ \Gamma & 1-\Gamma \end{pmatrix}$$

$$\exp\begin{pmatrix} \Gamma & \Gamma \\ -\Gamma & -\Gamma \end{pmatrix} = \begin{pmatrix} 1+\Gamma & \Gamma \\ -\Gamma & 1-\Gamma \end{pmatrix}$$

from equations (13)–(15), we get

$$e^{X\Gamma\tilde{X}} = [\det(1-\Gamma)]^{-1/2} : \exp\left[\frac{1}{2}\Lambda \begin{pmatrix} \Gamma/(1-\Gamma) & (1-\Gamma)^{-1} - 1 \\ (1-\Gamma)^{-1} - 1 & \Gamma/(1-\Gamma) \end{pmatrix} \tilde{\Lambda}\right] : \tag{18}$$

$$e^{P\Gamma\tilde{P}} = [\det(1-\Gamma)]^{-1/2} : \exp\left[\frac{1}{2}\Lambda \begin{pmatrix} -\Gamma/(1-\Gamma) & (1-\Gamma)^{-1} - 1 \\ (1-\Gamma)^{-1} - 1 & -\Gamma/(1-\Gamma) \end{pmatrix} \tilde{\Lambda}\right] :. \tag{19}$$

Equations (18) and (19) are equivalent to

$$e^{X\Gamma\tilde{X}} = [\det(1-\Gamma)]^{-1/2} : e^{X(1-\Gamma)^{-1}\tilde{X} - X\tilde{X}} : \tag{20}$$

$$e^{P\Gamma\tilde{P}} = [\det(1-\Gamma)]^{-1/2} : e^{P(1-\Gamma)^{-1}\tilde{P} - P\tilde{P}} :. \tag{21}$$

These are the same results as [9], but here the calculation is much simpler.

*(ii)* $e^{X\Gamma\tilde{P}}$. Here, it is not necessary for $\Gamma$ to be symmetric. As far as we know, this EQO had not yet been transformed into its normal product form. After transforming it into Fock space, we get

$$e^{X\Gamma\tilde{P}} = \exp\left(\frac{i}{2}\,\mathrm{tr}\,\Gamma\right)\exp\left[\frac{1}{2}\Lambda\cdot\frac{i}{2}\begin{pmatrix}-\Gamma+\tilde{\Gamma} & -\Gamma-\tilde{\Gamma}\\ -\Gamma-\tilde{\Gamma} & -\Gamma+\tilde{\Gamma}\end{pmatrix}\Sigma\tilde{\Lambda}\right]. \tag{22}$$

Denoting $S = \begin{pmatrix} I & -I \\ I & I\end{pmatrix}$, then $S^{-1} = \begin{pmatrix} I & I \\ -I & I\end{pmatrix}$ where $I$ is the $n$-dimensional unit matrix, it is easy to see that

$$\exp\left[\frac{i}{2}\begin{pmatrix}-\Gamma+\tilde{\Gamma} & -\Gamma-\tilde{\Gamma}\\ -\Gamma-\tilde{\Gamma} & -\Gamma+\tilde{\Gamma}\end{pmatrix}\right] = S^{-1}\exp\left[\frac{i}{2}\begin{pmatrix}\tilde{\Gamma} & 0\\ 0 & -\Gamma\end{pmatrix}\right]S \tag{23}$$

$$= \begin{pmatrix} e^{-i\Gamma/2}+e^{i\tilde{\Gamma}/2} & e^{-i\Gamma/2}-e^{i\tilde{\Gamma}/2}\\ e^{-i\Gamma/2}-e^{i\tilde{\Gamma}/2} & e^{-i\Gamma/2}+e^{i\tilde{\Gamma}/2}\end{pmatrix} \tag{24}$$

and, therefore, we get the result

$$e^{X\Gamma\tilde{P}} = e^{(i/2\,\mathrm{tr}\,\Gamma)}[\det(e^{i\Gamma/2}+e^{-i\tilde{\Gamma}/2})]^{-1/2}$$
$$\times :\exp\left[\frac{1}{2}\Lambda\begin{pmatrix} -(e^{i\Gamma/2}+e^{-i\tilde{\Gamma}/2})^{-1}(-e^{i\Gamma/2}+e^{-i\tilde{\Gamma}/2}) & (e^{i\Gamma/2}+e^{-i\tilde{\Gamma}/2})^{-1}-I\\ (e^{-i\Gamma/2}+e^{i\tilde{\Gamma}/2})^{-1}-I & (-e^{i\Gamma/2}+e^{-i\tilde{\Gamma}/2})(e^{i\Gamma/2}+e^{-i\tilde{\Gamma}/2})^{-1}\end{pmatrix}\tilde{\Lambda}\right]: \tag{25}$$

*(iii)* $e^{(\tilde{\alpha}a+a^{+}\beta)^2}$. Here, $\tilde{\alpha}$ and $\tilde{\beta}$ are $n$-dimensional complex vectors as $\tilde{\alpha} = (\alpha_1\alpha_2\cdots\alpha_n)$ $\tilde{\beta} = (\beta_1\beta_2\cdots\beta_n)$. Define operator $Q = \frac{\tilde{\alpha}}{\sqrt{\tilde{\alpha}\beta}}a$, $Q' = a^{+}\frac{\beta}{\sqrt{\tilde{\alpha}\beta}}$ and let $\gamma = (Q', Q)$, then we have $[\tilde{\gamma},\gamma] = \Sigma^{-1}$ and

$$e^{(\tilde{\alpha}a+a^{+}\beta)^2} = e^{\tilde{\alpha}\beta(Q+Q')^2}.$$

Noticing that

$$\exp\left[2\tilde{\alpha}\beta\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\right] = \begin{pmatrix}1+2\tilde{\alpha}\beta & -2\tilde{\alpha}\beta\\ 2\tilde{\alpha}\beta & 1-2\tilde{\alpha}\beta\end{pmatrix} \tag{26}$$

by equations (14) and (15) we get

$$\begin{aligned}\exp(\tilde{\alpha}a+a^{+}\beta)^2 &= \exp\left[\frac{1}{2}\gamma 2\tilde{\alpha}\beta\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\Sigma\tilde{\gamma}\right]\\ &= (1-2\tilde{\alpha}\beta)^{-1/2}:\exp\left[\frac{1}{2}\gamma\frac{2\tilde{\alpha}\beta}{1-2\tilde{\alpha}\beta}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\tilde{\gamma}\right]:\\ &= (1-2\tilde{\alpha}\beta)^{-1/2}:\exp\left[\frac{\tilde{\alpha}\beta}{(1-2\tilde{\alpha}\beta)}(Q+Q')^2\right]:\\ &= (1-2\tilde{\alpha}\beta)^{-1/2}:\exp[(1-2\tilde{\alpha}\beta)^{-1}(\tilde{\alpha}a+a^{+}\beta)^2]:.\end{aligned} \tag{27}$$

*(iv)* $e^{a_1^{+}a_2^{+}-a_1a_2}$. This boson operator has been transformed into normal product form in [3]. Now we deal with it again using equations (14) and (15). Obviously, it is equivalent to

$$\Omega = \exp\left[\frac{1}{2}\Lambda\begin{pmatrix}0 & -\sigma_1\\ -\sigma_1 & 0\end{pmatrix}\Sigma\tilde{\Lambda}\right] \tag{28}$$

where $\Lambda = (a_1^+, a_2^+, a_1, a_2)$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Noticing that

$$\exp\begin{pmatrix} 0 & -\sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (\mathrm{e}+\mathrm{e}^{-1})I & -(\mathrm{e}-\mathrm{e}^{-1})\sigma_1 \\ -(\mathrm{e}-\mathrm{e}^{-1})\sigma_1 & (\mathrm{e}+\mathrm{e}^{-1})I \end{pmatrix} \tag{29}$$

by equation (14) and (15), we obtain

$$\mathrm{e}^{a_1^+ a_2^+ - a_1 a_2} = (\mathrm{ch}1)^{-1} \cdot : \exp\left[\frac{1}{2}\Lambda\begin{pmatrix} \mathrm{th}1 \cdot \sigma_1 & [(\mathrm{ch}1)^{-1} - 1]I \\ [(\mathrm{ch}1)^{-1} - 1]I & -\mathrm{th}1 \cdot \sigma_1 \end{pmatrix}\tilde{\Lambda}\right] : . \tag{30}$$

In fact, after taking out the normal product sign, this is the same result as [3].

*(v)* $\mathrm{e}^{\tilde{a}\Gamma a} \cdot \mathrm{e}^{a^+ \Delta \tilde{a}^+}$. Here, $\Gamma$ and $\Delta$ are two $n \times n$ symmetric complex matrices. This operator is equivalent to

$$\Omega = \exp\left[\frac{1}{2}\Lambda\begin{pmatrix} 0 & 0 \\ 2\Gamma & 0 \end{pmatrix}\Sigma\tilde{\Lambda}\right]\exp\left[\frac{1}{2}\Lambda\begin{pmatrix} 0 & -2\Delta \\ 0 & 0 \end{pmatrix}\Sigma\tilde{\Lambda}\right]. \tag{31}$$

We can regard $\Omega$ as a LQT operator corresponding to the LQT matrix

$$M = \exp\begin{pmatrix} 0 & 0 \\ 2\Gamma & 0 \end{pmatrix}\exp\begin{pmatrix} 0 & -2\Delta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & -2\Delta \\ 2\Gamma & I - 4\Gamma\Delta \end{pmatrix}. \tag{32}$$

Therefore, according to (14), we get

$$\Omega = \frac{1}{\sqrt{\det(I - 4\Gamma\Delta)}} : \exp\left[\frac{1}{2}\Lambda\begin{pmatrix} 2(I - 4\Delta\Gamma)^{-1}\Delta & (I - 4\Delta\Gamma)^{-1} - I \\ (I - 4\Gamma\Delta)^{-1} - I & 2\Gamma(I - 4\Delta\Gamma)^{-1} \end{pmatrix}\tilde{\Lambda}\right] : . \tag{33}$$

Actually, this is just equivalent to the result in [9]. Here again, we can see that our approach is not only general but also direct and simple.

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## Appendix

For convenience, we denote $H(N) = \frac{1}{2}\Lambda N \Sigma \tilde{\Lambda}$ where $N \in C^{2n \times 2n}$. We shall calculate the expectation value of $\mathrm{e}^{tH(N)}$ in the vacuum state

$$\langle 0|\mathrm{e}^{tH(N)}|0\rangle.$$

According to equation (13), $N$ can be written in the following form without any loss of generality:

$$N = \begin{pmatrix} \alpha & -\beta \\ \gamma & -\tilde{\alpha} \end{pmatrix}$$

where $\gamma$ and $\beta$ are symmetric matrices. In addition, we assume that $e^{tN}$ has the following partitioned form:

$$e^{tN} = \begin{pmatrix} A(t) & D(t) \\ \tilde{B}(t) & \tilde{C}(t) \end{pmatrix}$$

where $A(t)$, $B(t)$, $C(t)$ and $D(t) \in C^{n\times n}$. Evidently, we have

$$\frac{d}{dt}\begin{pmatrix} A(t) & D(t) \\ \tilde{B}(t) & \tilde{C}(t) \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \gamma & -\tilde{\alpha} \end{pmatrix}\begin{pmatrix} A(t) & D(t) \\ \tilde{B}(t) & \tilde{C}(t) \end{pmatrix}. \tag{34}$$

If we denote

$$f(t) = \langle 0|e^{tH(N)}|0\rangle$$

by using the property of the vaccum state, we have

$$f'(t) = \tfrac{1}{2}\langle 0|(\tilde{a}\gamma a + \operatorname{tr}\tilde{\alpha})e^{tH(N)}|0\rangle. \tag{35}$$

From the transformation property of operator $e^{tH(N)}$ in equation (6), we know that

$$\begin{aligned} 0 = \langle 0|e^{tH(N)}a\tilde{a}|0\rangle &= \langle 0|[\tilde{D}(t)\tilde{a^+} + C(t)a][a^+D(t) + \tilde{a}\tilde{C}(t)]e^{tH(N)}|0\rangle \\ &= C(t)D(t)f(t) + C(t)\langle 0|a\tilde{a}e^{tH(N)}|0\rangle\tilde{C}(t). \end{aligned} \tag{36}$$

We assume $\det(C(t)) \neq 0$, then

$$\langle 0|a\tilde{a}e^{tH(N)}|0\rangle = -D(t)\tilde{C}(t)^{-1}f(t). \tag{37}$$

Therefore, we have

$$\langle 0|\tilde{a}\gamma a e^{tH(N)}|0\rangle = -f(t)\operatorname{tr}[\gamma D(t)\tilde{C}(t)^{-1}]. \tag{38}$$

Substituting equation (38) into equation (35), we get

$$f'(t) = f(t)\tfrac{1}{2}\operatorname{tr}[\tilde{\alpha} - \gamma D(t)\tilde{C}(t)^{-1}]. \tag{39}$$

From the relation between $N$ and $e^N$, we get

$$\frac{d\tilde{C}(t)}{dt} = \gamma D(t) - \tilde{\alpha}\tilde{C}(t).$$

Eventually, $f(t)$ satisfies

$$f'(t) = -\frac{1}{2}f(t)\operatorname{tr}\left[\frac{d\tilde{C}(t)}{dt}\tilde{C}(t)^{-1}\right].$$

Integrating this and using the condition $f(0) = 1$, we have

$$f(t) = \exp[-\tfrac{1}{2}\operatorname{tr}\ln\tilde{C}(t)] = [\det C(t)]^{-1/2}$$

and, therefore, equation (15) is proved.

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